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TURBULENT MOMENTUM DIFFUSIVITY WITHIN A CIRCULAR TUBE

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Abstract—The paper presents an analysis of turbulent shear flow in which Nikuradse's experimental variation of eddy diffusivity across a circular tube is predicted accurately from an harmonic mixing length theory. The harmonic form of characteristic length used in the analysis is derived from a theoretical consideration of the turbulent eddy structure.

NOMENCLATURE

B, constant of integration;
C, proportionality constant;
$f(\bar{r})$, non-dimensional correlation coefficient;
$g(\bar{r})$, non-dimensional correlation coefficient;
K, constant;
k, wave number;
L, outer scale of turbulence;
L_1, outer scale of turbulence in (R, θ) plane;
l_c, characteristic length of turbulence;
l_m, mixing length;
l_n, harmonic length;
R, radius of tube;
r_c, radius from centre of tube;
r, length of radial vector;
r_o, length of radial vector defining size and shape of critical eddy;
R_{ec}, critical Reynolds number;
U, velocity in direction of radius vector;
U_p, velocity in direction of radius vector;
U_n, velocity in direction normal to radius vector;
v, fluctuation velocity in tube radius direction;
w, fluctuation velocity in tube axial direction;
W, local mean velocity in tube axial (z) direction;
W_τ $= \left(\frac{\tau_s}{\rho}\right)^{1/2}$, friction velocity;
y, distance from tube wall measured along a radius;
τ, shear stress;
τ_s, wall shear stress;
ν, kinetic viscosity;
ϵ, eddy viscosity;
ρ, density;
θ, polar angular co-ordinate;
α, polar angular co-ordinate in (Z, R) plane;
γ, r_c/R.

Suffices
i, isotropic;
o, critical eddy dimension;
s, surface condition.

INTRODUCTION

AT THE moment research [1] is being directed at the intimate relations which exist between thermal and momentum diffusivities with the express purpose of specifying heat transfer rates more accurately. The emphasis in the present paper is laid on the accurate calculation of the momentum diffusivity for regions where the turbulent eddies are fully developed. An approach to the problem is made through a mixing length theory where the characteristic length is shown to depend on the microstructure of the turbulence through an harmonic mean length l_n defined by

$$\frac{1}{l_n} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{L_1}\,d\theta. \qquad (1)$$

The use of the harmonic mean length was first suggested to the author during the recent international heat transfer conference by Dr. Buleev of the U.S.S.R.

1. *The universal velocity profile for turbulent flow in round tubes*

It is instructive to re-examine the existing work for circular channels. Nikuradse's experiments show that near to the wall a universal velocity profile of the following form exists, Fig. 1.

$$\frac{W}{W_\tau} = \phi_1 \left(\frac{W_\tau y}{\nu}\right). \tag{2}$$

For laminar flow function ϕ_1 may be obtained directly from the Navier–Stokes equations.

$$\frac{W}{W_\tau} = \frac{yW_\tau}{\nu}. \tag{3}$$

In the turbulent region energy is extracted from the mean flow by the eddies. The experiments of Laufer [2] show that most of this energy is dissipated locally although a small percentage is diffused away from the wall towards the centre of the tube. Using this fact it may be supposed approximately that

$$\overline{wv} = \phi_2 \left(l_m, \frac{\mathrm{d}W}{\mathrm{d}y}\right) \tag{4}$$

where l_m is an unspecified mixing length which characterizes the turbulent fluctuations. From dimensionless analysis it follows that

$$\overline{wv} = l_m^2 \left(\frac{\mathrm{d}W}{\mathrm{d}y}\right)^2. \tag{5}$$

For high turbulence where viscosity may be neglected we have for regions near to the wall

$$\tau = \tau_s = \rho\overline{wv} = \rho l_m^2 \left(\frac{\mathrm{d}W}{\mathrm{d}y}\right)^2. \tag{6}$$

To integrate this equation it is usual to use Prandtl's mixing length given by

$$l_m = Ky. \tag{7}$$

Equation (2) now takes the form

$$\frac{W}{W_\tau} = \frac{1}{K} \log \left(\frac{yW_\tau}{\nu}\right) + B \tag{8}$$

where B is a constant and $K = 0{\cdot}4$.

In practice there is a smooth transition from the laminar to the turbulent region. Experiments show that the form of equation (8) is strictly justified only for regions near to the wall. This is to be expected because of the assumption of a constant shearing stress in the derivation of the equation. Away from the wall there is an increasing divergence between experimental results and the velocity gradient predicted by equation (8). For instance at the tube centre the predicted velocity gradient is given by

$$\left(\frac{\mathrm{d}W}{\mathrm{d}y}\right)_{y=R} = \frac{W_\tau}{KR} \tag{9}$$

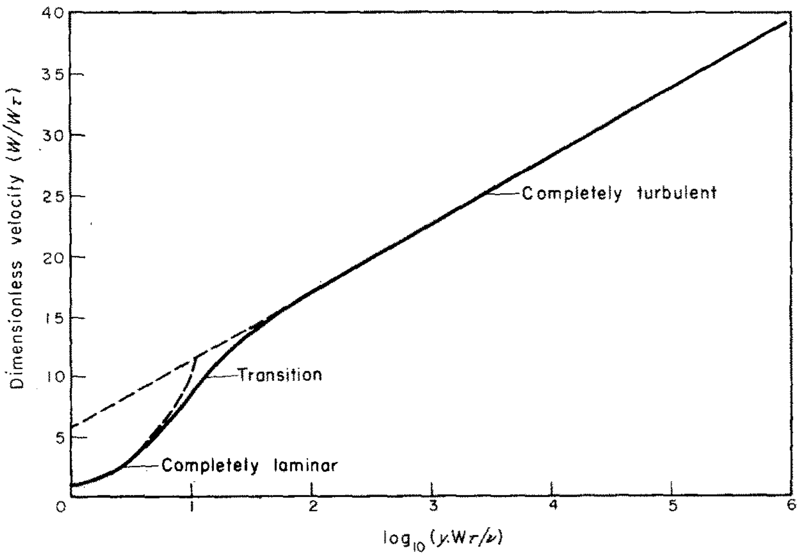

FIG. 1. Nikuradse plot universal velocity profile.

where physically it is zero. This divergence between the turbulent portion of the universal velocity profile represented by equation (8) and the experimental velocity profiles is emphasized when the eddy diffusivity is considered. The variation of eddy diffusivity is of prime importance in heat transfer calculations.

By definition

$$\tau = (\nu + \epsilon)\frac{\mathrm{d}W}{\mathrm{d}y}. \qquad (10)$$

In the turbulent region ν is small compared to ϵ and therefore from equations (8) and (10)

$$\epsilon = \frac{\tau}{\rho}\frac{\mathrm{d}y}{\mathrm{d}W} = \frac{\tau \cdot y}{2 \cdot 5 \rho W_\tau}. \qquad (11)$$

The distribution of shear stress for a tube is given by

$$\tau = \tau_S (1 - y/R) \qquad (12)$$

hence

$$\epsilon = \frac{\tau_S}{\rho} y(1 - y/R) \qquad (13)$$

or

$$\frac{\epsilon}{RW_\tau} = \frac{1}{2 \cdot 5}\left[\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2} - y/R\right)^2\right]. \qquad (14)$$

This is the equation of a parabola with its vertex at $\epsilon/RW_\tau = 0 \cdot 1$, $y/R = 0 \cdot 5$.

A comparison between this result based on the universal velocity profile and values calculated by Schlicting [3] directly from Nikuradse's experimental results is given in Fig. 2.

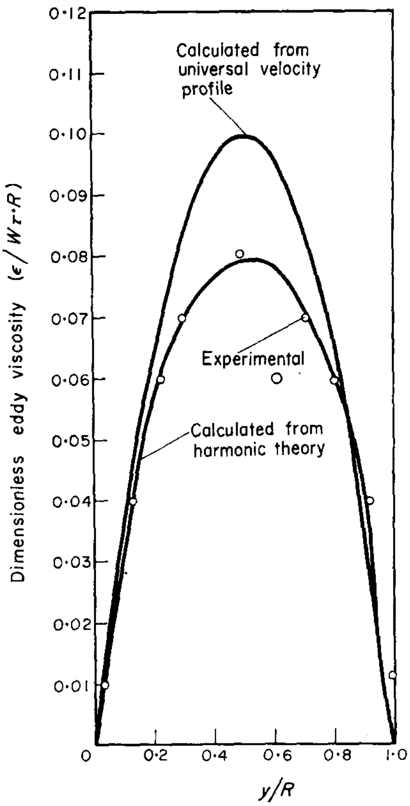

FIG. 2.

2. *Some conclusions from the mixing length theory*

For the general case in which viscosity is important we may write

$$\frac{\tau}{\rho} = -\nu\left(\frac{\mathrm{d}W}{\mathrm{d}r_c}\right) + \overline{wv} \qquad (15)$$

where $R = y + r_c$.

The mixing length and shear stress distribution may be introduced to yield

$$\frac{\tau_S}{\rho}(1 - y/R) = \nu\left(\frac{\mathrm{d}W}{\mathrm{d}y}\right) + l_m^2\left(\frac{\mathrm{d}W}{\mathrm{d}y}\right)^2, \qquad (16)$$

$$R \geqslant y \geqslant 0.$$

Solving (16) for $(\mathrm{d}W/\mathrm{d}y)$ we have

$$\frac{\mathrm{d}W}{\mathrm{d}y} = -\frac{\nu}{2l_m^2} + \left[\frac{\nu^2}{4l_m^4} + \frac{W_\tau^2}{l_m^2}(1 - y/R)\right]^{1/2} \qquad (17)$$

hence

$$\overline{wv} = l_m^2\left(\frac{\mathrm{d}W}{\mathrm{d}y}\right)^2 = \frac{\nu^2}{2l_m^2} + W_\tau^2(1 - y/R) - \frac{\nu}{l_m}\left[\frac{\nu^2}{4l_m^2} + W_\tau^2(1 - y/R)\right]^{1/2}. \qquad (18)$$

Certain deductions may now be made from equations (16, 17, 18).

(1) For $y \to R$

$\frac{\rho\overline{wv}}{\tau_S} \to (1 - y/R)$, providing $\frac{W_\tau l_m}{\nu}$ is large in this region.

(2) For $y = R$

$\overline{wv} = 0, \frac{\mathrm{d}W}{\mathrm{d}y} = 0.$

(3) From equation (18)

$$\frac{\rho\overline{wv}}{\tau_s} \to 0 \text{ as } y \to 0 \text{ providing } \frac{W_\tau l_m}{\nu} \to 0 \text{ as } y \to 0.$$

(4) At the wall we must have

$$\frac{\tau_s}{\rho} = W_\tau^2 = \nu\left(\frac{dW}{dy}\right)_s. \tag{19}$$

Substituting this value into equation (17)

$$\frac{dW}{dy} = -\frac{\nu}{2l_m^2} + \frac{\nu}{2l_m^2}\left[1 + \frac{4l_m^2}{\nu}(1 - y/R)\left(\frac{dW}{dy}\right)_s\right]^{1/2}.$$

Now for $|x| < 1$ the following approximation holds

$$(1 + x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \text{etc.} \tag{20}$$

Hence

$$\left.\begin{aligned} \frac{dW}{dy} &= -\frac{\nu}{2l_m^2} \\ &+ \frac{\nu}{2l_m^2}\left[1 + \frac{2l_m^2}{\nu}(1 - y/R)\left(\frac{dW}{dy}\right)_s\right. \\ &\left.- \frac{2l_m^4}{\nu^2}(1 - y/R)^2\left(\frac{dW}{dy}\right)_s^2 + \dots\right] \\ &= \left[(1 - y/R)\left(\frac{dW}{dy}\right)_s\right. \\ &\left.- \frac{l_m^2}{\nu}(1 - y/R)^2\left(\frac{dW}{dy}\right)_s^2 + \dots\right]. \end{aligned}\right\} \tag{21}$$

Then providing $l_m \to 0$ as $y \to 0$ we must have that

$$\frac{dW}{dy} \to \left(\frac{dW}{dy}\right)_s \text{ as } y \to 0.$$

It is insufficient however, to show that the mixing length theory is consistent with the fluid boundary conditions but it must be further demonstrated that the characteristic length l_m has a physically realistic value within the tube geometry. Fortunately, Nikuradse has calculated the true variation of l_m from the experimental velocity profiles [3, p. 511] and this is shown in Fig. 3. The errors in the predicted eddy diffusivity described in Section 1 are therefore attributed to the wrong choice of a mixing length and to the neglect of the true shearing stress variation in the derivation of equation (8).

3. *Derivation of a characteristic length for turbulent flow in a duct*

The turbulent motion of a fluid is made up of eddies and fluctuations of all orders which grow, exist for some period of time in a state of virtual equilibrium, and then decay. In their publications on the theory of turbulence both Batchelor [4, p. 105] and Townsend [5, p. 43] indicate that the characteristic length for the turbulent fluctuations in an infinite turbulent velocity field may be written.

$$l_c = \int_0^\infty f(\bar{r})\,dr \tag{22}$$

where $f(\bar{r})$ is a non-dimensional velocity correlation coefficient defined in Fig. 4. An approximate shape for $f(\bar{r})$ is illustrated by Batchelor in his book [4, p. 48]. In the formation of any theory the first problem is to construct a suitable mathematical model of the turbulence which yields a satisfactory form for $f(\bar{r})$. Of the existing theories the most widely accepted model of the turbulent structure is that presented by Kolmogoroff [6]. Within the spectrum of all the eddies Kolmogoroff supposes that there are some which neither grow nor decay with time. The whole of the energy subtracted from the mean flow is then assumed to be dissipated in these critically sized eddies. Because the classical Kolmogoroff analysis led to some difficulty a new approach was made, which however, retained the idea of a critical sized eddy in the turbulent structure.

Consider a fluid flow characterized by the values of the kinematic viscosity ν, the characteristic velocity scale U, and the characteristic length scale l. This flow is stable only where the local Reynolds number does not exceed a certain value R_{ec}. With increase of the local Reynolds number beyond the critical value large eddies become unstable and break down into smaller eddies transferring most of their kinetic energy in the process. The mean velocity at which an eddy begins to break down is then given approximately by

$$U = \frac{R_{ec}\nu}{l}. \tag{23}$$

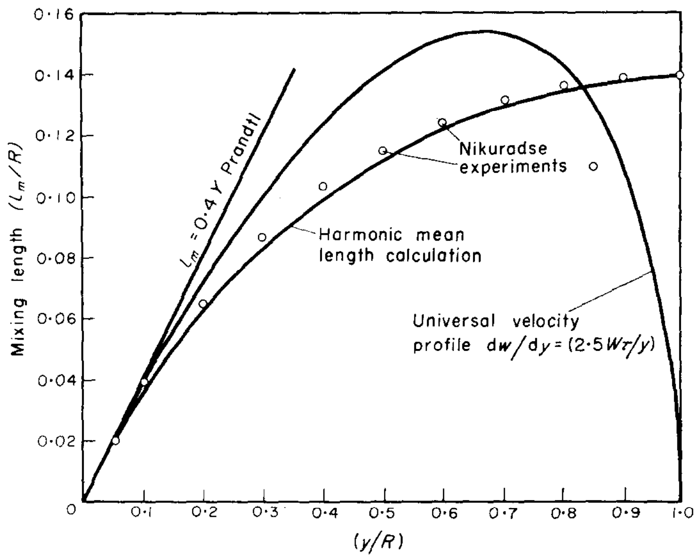

FIG. 3.

The purpose of this paper is to deduce results which may be applied to non-isotropic shear flow. Consider further therefore, the velocity correlation which exists during the period of equilibrium between a non-isotropic critical eddy of mean dimension $\bar{r}_o$ and another non-isotropic eddy of mean dimension $\bar{r}$, each of these eddies being centred at $r = 0$. Strictly speaking no finite boundaries exist to an eddy since the properties of the fluid must be continuous. When eddies of some finite size $\bar{r}$ are described by their mean property it implies that a discontinuity has been introduced into the analysis by the lumping procedure.

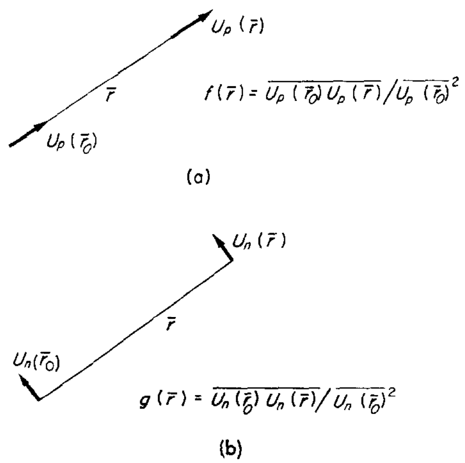

FIG. 4. Definition of correlation functions.

To obtain a solution the following assumptions are made.

(1) That eddies having the same origin ($r = 0$) begin to decay at the same critical Reynolds number.

(2) That critical eddies are deformed in a given direction in proportion to the outer scale of turbulence L in that direction, e.g. if L is constant in all directions then the eddies become isotropic.

(3) Within the critical sized eddy each portion of the fluid is attributed the mean flow characteristics of that eddy.

Under these conditions the equations for the correlation coefficient may be written:

For $\bar{L} > \bar{r} > \bar{r}_o$

$$\overline{U_p(\bar{r})U_p(\bar{r}_o)} = \frac{R_{ec}^2\nu^2}{\bar{r}_o\bar{r}} \tag{24}$$

$$f(\bar{r}) = \frac{(R_{ec}^2\nu^2)/(\bar{r}_o\bar{r})}{(R_{ec}^2\nu^2)/(\bar{r}_o^2)} = \frac{\bar{r}_o}{\bar{r}} \tag{25}$$

and for $\bar{r}_o \geqslant \bar{r} \geqslant 0$

$$f(\bar{r}) = \frac{(R_{ec}^2\nu^2)/(\bar{r}_o^2)}{(R_{ec}^2\nu^2)/(\bar{r}_o^2)} = 1. \tag{26}$$

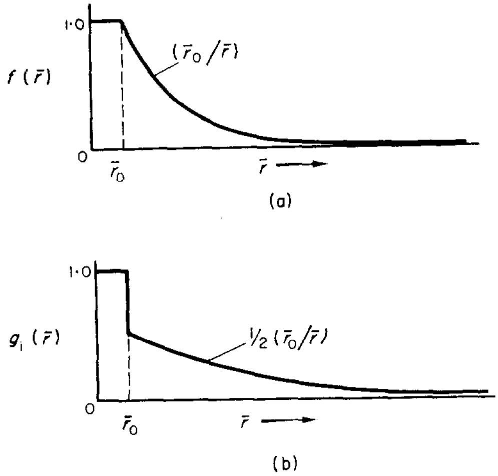

FIG. 5. Theoretical form of correlation functions.

The approximating shape of the correlation coefficient $f(\bar{r})$ is then shown in Fig. 5. For large values of $\bar{r}$ the approximation to the dimensionless correlation coefficient tends to zero.

An interesting fact now emerges when the complementary correlation function $g_i(\bar{r})$ is considered Fig. 5. For isotropic turbulence Batchelor [4, p. 46] has shown that $g_i(\bar{r})$ is related to $f_i(\bar{r})$ by the condition of continuity, namely

$$g_i(\bar{r}) = f_i(\bar{r}) + \tfrac{1}{2}\,\bar{r}\,\frac{\partial f_i(\bar{r})}{\partial \bar{r}}. \tag{27}$$

Hence for $\bar{L} > \bar{r} > \bar{r}_o$

$$g_i(\bar{r}) = \tfrac{1}{2} f_i(\bar{r}) \tag{28}$$

and for $\bar{r}_o \geqslant \bar{r} \geqslant 0$

$$g_i(\bar{r}) = 1. \tag{29}$$

Results (28) and (29) show for isotropic turbulence that $g_i(\bar{r})$ is discontinuous at $\bar{r} = \bar{r}_o$, Fig. 5. In reality the slope of $f_i(\bar{r})$ must be continuous at $\bar{r} = \bar{r}_o$, leading to a continuous but quickly changing value for $g_i(\bar{r})$. The mathematical model therefore creates a region in which there is a drastic change in the circulation velocity at $\bar{r} = \bar{r}_o$ which permits dissipation of large quantities of energy by viscous action. A form for the macroscopic characteristic length l_c may now be determined for a bounded turbulent velocity field.

$$l_c = \int_0^{\bar{L}} f(\bar{r})\,d\bar{r} = \int_0^{\bar{r}_o} f(\bar{r})\,d\bar{r} + \int_{\bar{r}_o}^{\bar{L}} f(\bar{r})\,d\bar{r}$$

$$= \bar{r}_o + \bar{r}_o \log_e \left(\frac{\bar{L}}{\bar{r}_o}\right)$$

or

$$l_c = \frac{1 + \log_e(\bar{L}/\bar{r}_o)}{1/\bar{r}_o}. \tag{30}$$

In this book on the structure of turbulent shear flow Townsend [5, p. 12] has shown that the wave number k of an eddy of size r is approximately given by

$$k = \frac{1}{r}. \tag{31}$$

The mean value of the wave number associated with the mean size of the critical eddy must therefore be written

$$\bar{k}_o = \frac{1}{\bar{r}_o} = \frac{\int (1/r_o)\,d\eta}{\int d\eta} \tag{32}$$

where η is the solid angle. The element of solid angle $d\eta$ is defined by $d\eta = \sin\alpha\, d\alpha\, d\theta$ where r_o, α, θ are the polar co-ordinates defining the boundaries of the critical eddy.

Making use of the second assumption we may also write

$$r_o = CL \tag{33}$$

where C is a constant and L the outer scale of turbulence in any direction. Equation (3) for the characteristic length l_c then becomes

$$l_c = \frac{[1 + \log(1/C)]\int d\eta}{\int (1/r_o)\,d\eta}. \tag{34}$$

For the special case of a parallel sided duct of constant cross-sectional area, the eddies become symmetrical in the axial direction and equation (34) reduces to

$$l_c = \frac{(4/\pi)\,[(1 + \log(1/C))/(1/C)]}{(1/2\pi)\int_0^{2\pi} (1/L_1)\,d\theta} \tag{35}$$

where L_1 is measured in the (R, θ) plane, i.e. $L_1 = L\sin\alpha$.

It remains to show that the constructed mathematical model is consistent with the experimental evidence.

4. *Experimental verification of the turbulence structure model*

For a circular tube we have

$$L_1^2 + 2r_c L_1 \cos\theta - (R^2 - r_c^2) = 0. \tag{36}$$

Hence

$$\frac{R}{l_c} = \frac{1}{2\pi C_1}\int_0^{2\pi} \frac{d\theta}{-\gamma\cos\theta + (1-\gamma^2\sin^2\theta)^{1/2}} \tag{37}$$

where

$$C_1 = \frac{4}{\pi}\left[\frac{1+\log_e(1/C)}{(1/C)}\right] \text{ and } \gamma = r_{c/R}.$$

Therefore

$$\frac{R}{l_c} = \frac{2}{\pi C_1(1-\gamma^2)}\int_0^{\pi/2} (1-\gamma^2\sin^2\theta)^{1/2}\, d\theta$$

or

$$\frac{l_c}{R} = \frac{(\pi C_1/2)(1-\gamma^2)}{\int_0^{\pi/2}(1-\gamma^2\sin^2\theta)^{1/2}\,d\theta}. \tag{38}$$

If the characteristic length is to be associated with the mixing length then from Nikuradse's results, Fig. 3, we must have that

$$\frac{1+\log_e(1/C)}{(1/C)} = 0{\cdot}14\left(\frac{\pi}{4}\right). \tag{39}$$

The solution of equation (39) is presented in graphical form in Fig. 7. From this the cut off value of $f(\bar{r})$ is determined, i.e. $L = 44{\cdot}5 r_0$ giving $f(\bar{r})_{\bar{r}=\bar{L}} = 1/44{\cdot}5 = 0{\cdot}0225$ which is an acceptable value.

Evaluating the elliptic integral in equation (38) the following results are obtained (Table 1).

Calculated mixing lengths

(a) For Prandtl's assumption of $l_m = 0{\cdot}4y$.
(b) For the universal velocity profile.
(c) For l_c.

are compared to Nikuradse's experimental values in Fig. 3. It is seen that the agreement of l_c with Nikuradse's curve is almost perfect.

The experimental variation of eddy diffusivity may also be predicted using the calculated value of l_c for l_m.

Table 1

r_c/R	l_c/R
0·0	0·14
0·1	0·139
0·2	0·136
0·3	0·130
0·4	0·123
0·5	0·112
0·6	0·098
0·7	0·0826
0·8	0·0616
0·9	0·0338
0·95	0·020
1·0	0·0

For $\epsilon \gg \nu$

$$\frac{\tau}{\rho} = \tau_s(1 - y/R) = l_m^2\left(\frac{dW}{dy}\right)^2 = l_c^2\left(\frac{dW}{dy}\right)^2.$$

Hence

$$\frac{\epsilon}{RW_\tau} = \frac{l_c}{R}(1-y/R)^{1/2}, \text{ where } \epsilon = l_c^2\left(\frac{dW}{dy}\right).$$

Substituting the calculated values for l_c the following results were derived (Table 2).

These calculated values of the eddy viscosity

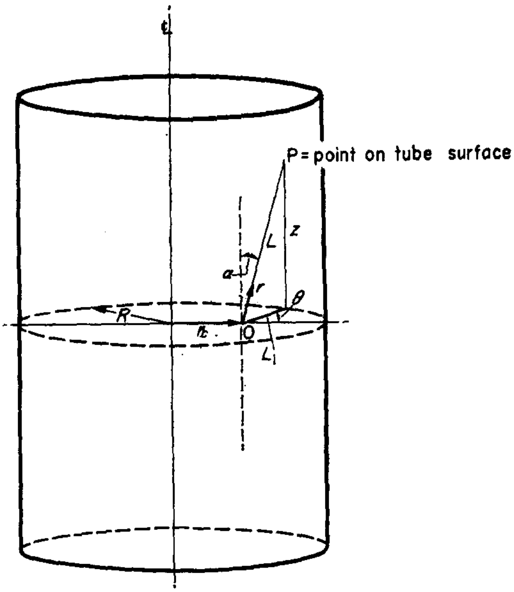

FIG. 6. Diagram of co-ordinates.

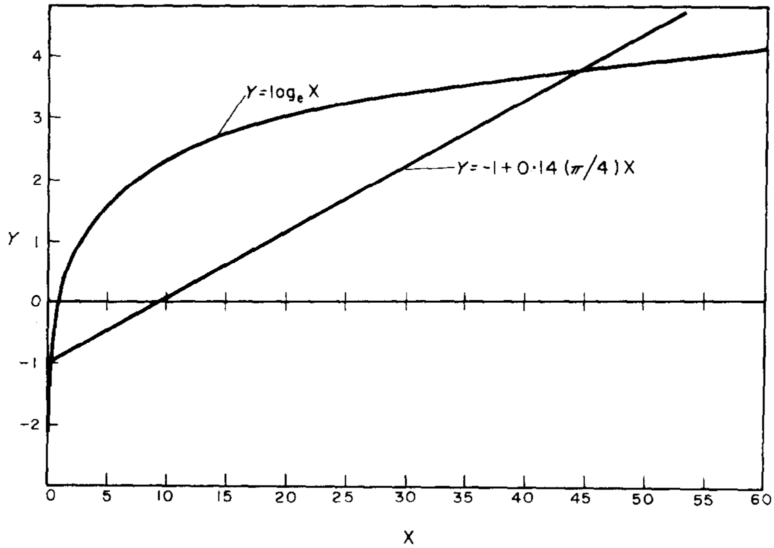

FIG. 7.

Table 2

r_c/R	ϵ/RW_τ
0	0·0
0·1	0·0440
0·2	0·0609
0·3	0·0714
0·4	0·0780
0·5	0·0793
0·6	0·0760
0·7	0·0693
0·8	0·0551
0·9	0·0340
0·95	0·0195
1·0	0·0

are compared to the experimental results in Fig. 2 and the agreement is again almost perfect.

CONCLUSIONS

A tentative mathematical model has been constructed for the turbulent motion of an incompressible fluid in shear flow within a duct. In the analysis it is assumed that all the energy extracted from the mean flow is dissipated locally. This assumption is not completely true and accounts for the small discrepancy between calculated and experimental diffusivity at the tube centre as illustrated in Fig. 2. Nevertheless, the model gives excellent agreement with experiment for a circular tube. Further analytical research is required before the turbulent structure can be specified.

(a) In the transition region of the boundary layer, [7].

(b) In the outer edge of a growing boundary where the so called "Law of the Wake" applies, [8].

An effort is now being made towards an understanding of these phenomena.

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Résumé—L'article présente une étude de l'écoulement turbulent dans lequel la variation de la diffusivité turbulente de Nikuradse dans un tube circulaire est calculée avec précision à partir d'une théorie harmonique de la longueur de mélange. La forme harmonique de la longueur caractéristique utilisée dans l'étude est déduite d'une considération théorique de la structure des tourbillons.

Zusammenfassung—Die Arbeit stellt eine Analyse dar der turbulenten Scherströmung, in der Nikuradses experimentelle Variation des turbulenten Austausches in einem Rohr mit Kreisquerschnitt mit einer harmonischen Mischlängentheorie genau bestimmbar wird. Die harmonische Form der Analyse benutzten charakteristischen Länge wird aus einer theoretischen Betrachtung der turbulenten Wirbelstruktur gewonnen.

Аннотация—В статье дается анализ турбулентного вихревого течения, в котором по экспериментальным данным Никурадзе точно рассчитано изменение коэффициента турбулентной диффузии в круглой трубе с помощью теории гармонической длины смешения.

Гармоническая форма характерной длины, используемая в анализе, выводятся из теоретического рассмотрения структуры турбулентных вихрей.